

DEGENERATE HESSIAN STRUCTURES ON RADIANT MANIFOLDS

M. Á. GARCÍA-ARIZA

ABSTRACT. A manifold M is said to be radiant if it is endowed with a distinguished atlas whose coordinate changes are locally linear transformations. Such manifolds are thereby furnished with a flat connection $\bar{\nabla}$ and a radiant vector field ρ whose covariant derivative is the identity mapping. A degenerate Hessian metric on M is a symmetric, positive semi-definite tensor field g of type $(0, 2)$ that can locally be written as the covariant Hessian of a function, called local potential, and is such that the canonical mapping \flat that it induces from the tangent bundle to the cotangent bundle of M is non-injective. A function on M is said to be extensive if its Lie derivative with respect to ρ is the function itself. In this paper, it is shown that the local potentials of g are extensive if and only if $\flat(\rho) = 0$. It is also shown that degenerate Hessian manifolds always contain Hessian submanifolds, which are proved to be embedded. Their slice coordinates are elements of the aforementioned distinguished atlas. Applications to classical equilibrium thermodynamics are discussed.

1. INTRODUCTION

Let M be a finite-dimensional affine manifold, *i. e.*, a smooth manifold endowed with a flat affine connection $\bar{\nabla}$. A *degenerate Hessian metric* on M is a symmetric, positive semi-definite tensor field g of type $(0, 2)$ on M that is locally written as the Hessian (with respect to $\bar{\nabla}$) of a local smooth function—called henceforth *potential* of the degenerate Hessian metric—and induces a non-injective mapping \flat from the tangent bundle of M , TM , to its cotangent bundle T^*M , defined by

$$\flat(X)(Y) = g(X, Y),$$

for all vector fields X and Y defined on M . The pair $(\bar{\nabla}, g)$ is called a *degenerate Hessian structure* on M , and the latter is called a *degenerate Hessian manifold*.

Degenerate Hessian structures constitute a generalization of Hessian structures [12]. Their study is motivated by the geometry of equilibrium thermodynamics. As first pointed out by Weinhold [14], an interesting consequence of the laws of thermodynamics (*viz.*, the First and Second Laws, and the Principle of Maximum Entropy) is that the space of equilibrium states of any classical thermodynamic system is naturally endowed with a degenerate Hessian metric, whose components with respect to its entropy representation (a physically relevant coordinate chart on this manifold) are given by the Hessian matrix of the negative of the entropy of the system [8]. With the aid of this tensor, many of the well-known equations of equilibrium thermodynamics can be given a vector-geometric interpretation [13].

A distinguishing feature of the degenerate Hessian structures appearing in equilibrium thermodynamics is that their potentials are *extensive functions* (up to an additive constant). In elementary terms, this means that they are degree-one homogeneous functions of the entropy representation. A suitable global definition of

extensivity on affine manifolds is introduced in Section 2 by means of a radiant vector field [5]. As is shown, extensive potentials (up to an additive constant) are necessary and sufficient for a Hessian metric to be degenerate, as occurs in thermodynamics. The choice of a particular radiant vector (or, equivalently, of an entropy representation in thermodynamics) endows an affine manifold with a radiant structure [6]. Conversely, any radiant manifold is provided with a well-defined concept of extensivity. Under this approach, spaces of equilibrium states in thermodynamics are radiant manifolds endowed with a degenerate Hessian metric having the radiant vector field as null vector.

When restricted to appropriate Hessian submanifolds of spaces of equilibrium states (defined by constant volume of the system), degenerate Hessian metrics have become relevant in the framework of thermodynamic fluctuation theory, owing to Ruppeiner's so-called *interaction hypothesis*: their scalar curvature yields valuable microscopic information, unattainable from macroscopic thermodynamics by any other known means [9, 10]. In Section 3, the Hessian submanifolds of degenerate Hessian manifolds are studied. A mathematical generalization of Ruppeiner's approach is provided.

Section 4 is devoted to concluding remarks.

2. EXTENSIVE DIFFERENTIAL FORMS ON RADIANT MANIFOLDS

In what follows, $(\bar{\nabla}, g)$ denotes a degenerate Hessian structure on an n -dimensional smooth manifold M , with $n \in \mathbb{N}$; $\mathfrak{X}(M)$ and $\bigwedge^k(M)$ denote the set of vector fields and differential k -forms on M , for $k \in \mathbb{N} \cup \{0\}$, respectively. Einstein's summation convention over repeated indices is used. For any vector field $X \in \mathfrak{X}(M)$, $\flat(X)$ will be denoted by X^\flat .

As mentioned above, the concept of extensivity plays a significant role in equilibrium thermodynamics: entropy, which is a global potential for the degenerate Hessian metric on the space of equilibrium states is commonly required to be extensive [2, p. 29] up to an additive constant (corresponding to the value of entropy on a fixed reference state). This is closely related to the fact that $\ker \flat$ is nontrivial, as the following examples illustrates.

Example 1. Let \mathcal{E}_{ig} denote the 3-dimensional space of equilibrium states of an ideal gas [2]. A (global) entropy representation on this manifold is given by the functions U , V , and N , which denote the internal energy, volume, and number of particles of the system, respectively. Define a global flat affine connection $\bar{\nabla}$ on \mathcal{E}_{ig} by

$$\bar{\nabla}_i dx^j = 0$$

for each $i, j \in \{1, 2, 3\}$, where $\bar{\nabla}_i := \bar{\nabla}_{\partial/\partial x^i}$ and $(x^1, x^2, x^3) := (U, V, N)$.

In order to construct the degenerate Hessian metrics g , consider the entropy S of this system, which in terms of its entropy representation is given by

$$(1) \quad S = NR \ln \left(K V U^c N^{-(c+1)} \right) + S_0,$$

with $c, K, R, S_0 \in \mathbb{R}$ and $c, R > 0$. The matrix representation of $g := \bar{\nabla}d(-S)$ with respect to the chart (U, V, N) reads

$$(g_{ij}) = R \begin{pmatrix} c \frac{N}{U^2} & 0 & -\frac{c}{U} \\ 0 & \frac{N}{V^2} & -\frac{1}{V} \\ -\frac{c}{U} & -\frac{1}{V} & \frac{c+1}{N} \end{pmatrix},$$

which is also the matrix representation of \flat in these coordinates.

It can readily be seen that $\det \flat = 0$ (*i. e.*, g is degenerate) and $\ker \flat$ is spanned by

$$(2) \quad E := U \frac{\partial}{\partial U} + V \frac{\partial}{\partial V} + N \frac{\partial}{\partial N}.$$

Thus, the pair $(\bar{\nabla}, g)$ forms a degenerate Hessian structure on \mathcal{E}_{ig} .

Example 2. The realization of a space of equilibrium states as a degenerate Hessian manifold is commonly straightforward when a fundamental equation of state (an explicit expression for entropy in terms of the entropy representation of the system) is provided. To illustrate this assertion, consider the equation

$$U = NRT_0 \exp \left(\frac{S}{NR} + \frac{I^2}{N^2 I_0^2} \right),$$

which is the fundamental equation of an ideal paramagnetic solid [2, p. 83], where T_0 and I_0 are positive constants. In this case, U , S , I , and N represent the internal energy, entropy, magnetic moment and number of particles of the system, respectively.

It can readily be seen that S is a degree-one homogeneous function of (U, I, N) , whence this triad can be considered a global entropy representation on a 3-dimensional space of equilibrium states \mathcal{E}_{ip} . Again, define $\bar{\nabla}$ by demanding that (U, I, N) be an affine coordinate chart thereof (this is, a coordinate chart on which the Christoffel symbols of $\bar{\nabla}$ vanish). The matrix representation of $g = \bar{\nabla} d(-S)$ with respect to the entropy representation is given by

$$(g_{ij}) = R \begin{pmatrix} \frac{N}{U^2} & 0 & -\frac{1}{U} \\ 0 & \frac{2}{NI_0^2} & -\frac{2I}{N^2 I_0^2} \\ -\frac{1}{U} & -\frac{2I}{N^2 I_0^2} & \frac{1}{N} \left(1 + \frac{2I^2}{I_0^2 N^2} \right) \end{pmatrix}.$$

Like in the previous example, $\ker \flat$ is spanned by the Euler vector field

$$E = U \frac{\partial}{\partial U} + I \frac{\partial}{\partial I} + N \frac{\partial}{\partial N}.$$

Since g is positive semi-definite, $(\bar{\nabla}, g)$ is a degenerate Hessian structure on \mathcal{E}_{ip} .

The previous examples suggest that the particular form of the null vector field of \flat and the fact that S is extensive (up to an additive constant) are related to each other. To display this relationship precisely, a global idea of *extensivity* on affine manifolds is introduced, upon observing that in both cases, $\bar{\nabla} E$ is the identity mapping (when considered a vector-valued 1-form). This means that E is a radiant vector field [5] in each case. Using the fact that M is affine (*i. e.*, it is endowed with an affine flat connection), extensivity thereon may be defined by means of a radiant vector.

A vector field $\rho \in \mathfrak{X}(M)$ is called radiant whenever it satisfies

$$(3) \quad \bar{\nabla}_X \rho = X,$$

for all $X \in \mathfrak{X}(M)$. (The existence of a global radiant vector field on M depends on the existence of fixed points of the affine holonomy on M [5].) As is well known, a radiant vector field can be locally written as in Eq. (2). Namely, if ρ is a radiant

vector defined on an open subset \mathcal{U} of M , and (\mathcal{V}, ϕ) , is an affine coordinate chart contained in \mathcal{U} , with $\phi = (x^1, \dots, x^n)$, Eq. (3) yields

$$\rho|_{\mathcal{V}} = (x^i + c^i) \frac{\partial}{\partial x^i}.$$

Upon defining a new set of affine coordinates (x'^1, \dots, x'^n) on \mathcal{V} by $x'^i = x^i - c^i$ for all $i \in \{1, \dots, n\}$, the assertion follows.

Notice that if (\mathcal{V}_1, ϕ_1) and (\mathcal{V}_2, ϕ_2) are two overlapping coordinate charts where ρ has the form (2), then $\phi_2 \circ \phi_1^{-1}$ is a linear transformation. Hence, the choice of a particular radiant vector endows M with a $\text{GL}(n)$ structure, making it a radiant manifold [6]. In what follows, it shall be supposed that M is endowed with a global radiant vector field ρ . In other words, (M, ρ) will be assumed to be a radiant manifold.

Remark. Since radiant vector fields are determined up to a $\bar{\nabla}$ -parallel vector field, the concept of extensivity is subordinated to the choice of a particular radiant vector field on the space of equilibrium states of a thermodynamic system. Once chosen, this vector endows the manifold with a $\text{GL}(n)$ structure. The entropy representation of the system is only one coordinate chart of the distinguished atlas that is thereby induced. This suggests that a physically meaningful scalar should be $\text{GL}(n)$ -invariant.

On the other hand, any element of the atlas on M consisting of charts whose transition functions are elements of $\text{GL}(n)$ is the local analogue of the entropy representation of the space of equilibrium states of a thermodynamic system. For this reason, such coordinate charts will be referred to henceforth as *thermodynamic representations*.

Following the examples above, the existence of a radiant vector ρ on M allows a coordinate-free definition of extensive differential forms (cf. Ref. [1]).

Definition 1. A local k -form ω defined on M is said to be *extensive* if

$$\mathcal{L}_\rho \omega = \omega.$$

As mentioned before, degenerate Hessian structures occurring in thermodynamics have potentials that are extensive up to an additive constant (equivalently, the differentials of these potentials are extensive). This particular feature is what renders the former degenerate.

Theorem 1. *The differential of any potential of g is extensive if and only if $\rho^\flat = 0$.*

This result follows readily upon noticing that

$$X^\flat = \mathcal{L}_X d\Phi - d\Phi \circ \bar{\nabla} X,$$

for any local vector field X that shares domain with a local potential Φ of g .

Remark. The equation $\rho^\flat = 0$ is a coordinate-free version of the well-known *Gibbs-Duhem* equation.

Example 3. The realization of the space of equilibrium states of a system may not be as straightforward as in Example 2 when considering more *exotic* systems, like black holes. It is well-known that these are thermodynamic systems, but their

particular physical features render them special. For instance, the entropy of Kerr-Newman black holes is given by

$$S = \frac{1}{4} \left(M^2 + M^2 \sqrt{1 - \frac{Q^2}{M^2} - \frac{J^2}{M^4} - \frac{Q^2}{2}} \right),$$

where S represents the entropy of a given black hole, with mass M , charge Q and angular momentum J . According to the no-hair theorem, these three parameters characterize a black hole. Hence, these can be regarded as a global coordinate chart on a 3-dimensional space of equilibrium states \mathcal{E}_{KN} . The choice of an affine structure (*i. e.*, a flat affine connection) hereon presents the following problem: S is not a degree-one homogeneous function of M , Q and J , which means that the Hessian of $-S$ will not be semi-definite. This is generally not considered a problem, since self gravitating systems are in general regarded as quasi-homogeneous systems [1]. Nonetheless, a different approach may be taken, demanding that these systems possess the same mathematical structure as classical ones. Such an approach suggests that M^2 , Q^2 , and J should define a radiant structure on \mathcal{E}_{KN} [3].

If $\ker \flat$ is spanned by the radiant vector on M (as is the case in thermodynamics), any submanifold transversal to this vector is Riemannian. The scalar curvature of one of these submanifolds is claimed to play an important physical role, according to Ruppeiner's interaction hypothesis. In the next section, it is shown that Riemannian submanifolds always exist, and that Ruppeiner's submanifold belongs to a particular class of the latter.

3. HESSIAN SUBMANIFOLDS OF DEGENERATE HESSIAN MANIFOLDS

The existence of Riemannian submanifolds of M is guaranteed by the fact that the null vectors of \flat define an integrable distribution.

Indeed, since $\bar{\nabla}$ is symmetric, one has that

$$[X, Y]^\flat(Z) = \bar{\nabla}_X(Y^\flat(Z)) - \bar{\nabla}_X g(Y, Z) - Y^\flat(\bar{\nabla}_X Z) - \bar{\nabla}_Y(X^\flat(Z)) + \bar{\nabla}_Y g(X, Z) + X^\flat(\bar{\nabla}_Y Z).$$

Besides, degenerate Hessian structures satisfy the Codazzi equation:

$$(4) \quad \bar{\nabla}_X g(Y, Z) = \bar{\nabla}_Y g(X, Z),$$

for all $X, Y, Z \in \mathfrak{X}(M)$, whence

$$[X, Y]^\flat(Z) = \bar{\nabla}_X(Y^\flat(Z)) - Y^\flat(\bar{\nabla}_X Z) - \bar{\nabla}_Y(X^\flat(Z)) + X^\flat(\bar{\nabla}_Y Z).$$

Thus, $X^\flat = Y^\flat = 0$ implies $[X, Y]^\flat = 0$. This means that the distribution defined by the null vectors of \flat is involutive, and hence, completely integrable around every point of M . Consequently, any manifold that is transversal to an integral manifold of the latter is Riemannian.

Observe that, since locally $g = \bar{\nabla} d\Phi$ on M , then $\iota^* g = (\iota^* \bar{\nabla}) d(\iota^* \Phi)$ locally, on any Riemannian submanifold $\iota : N \hookrightarrow M$. If $\iota^* \bar{\nabla}$ is flat, then it forms together with $\iota^* g$ a Hessian structure on N .

Let $\iota : N \hookrightarrow M$ denote an r -dimensional Hessian submanifold of M , (notice that $r \leq \text{rank } \flat$). Since $(N, \iota^* \bar{\nabla})$ is affine, there exists a coordinate chart $(\mathcal{V}, (z^1, \dots, z^r))$ around any point $p \in N$, such that

$$(\iota^* \bar{\nabla})_\alpha dz^\beta = 0,$$

for all $\alpha, \beta \in \{1, \dots, r\}$. Let $(\mathcal{U}, (x^1, \dots, x^n))$ be a thermodynamic representation around $\iota(p)$; by definition,

$$(\iota^* \bar{\nabla})_\alpha \iota^*(dx^j) = \iota^* \left(\frac{\partial \iota^* x^k}{\partial z^\alpha} \bar{\nabla}_k dx^j \right),$$

for all $\alpha \in \{1, \dots, r\}$ and $j \in \{1, \dots, n\}$, which vanishes due to the connection's being flat. Alternatively,

$$(\iota^* \bar{\nabla})_\alpha \iota^*(dx^j) = \frac{\partial^2 \iota^* x^j}{\partial z^\alpha \partial z^\beta} dz^\beta,$$

whence

$$(5) \quad \frac{\partial \iota^* x^j}{\partial z^\alpha} = a_\alpha^j,$$

with $a_\alpha^j \in \mathbb{R}$, for all $j \in \{1, \dots, n\}$ and $\alpha \in \{1, \dots, r\}$. Since ι is an immersion, $\text{rank } \iota = r$, whence there exists an invertible $n \times n$ real matrix (b_i^j) such that $b_i^j a_\alpha^i = 0$, for each $j \in \{r+1, \dots, n\}$ and $\alpha \in \{1, \dots, r\}$. Therefore, the thermodynamic representation $(\mathcal{U}, (\tilde{x}^1, \dots, \tilde{x}^n))$ given by

$$\tilde{x}^k := b_i^k x^i$$

is a slice coordinate chart of N . In other words, the following proposition has been proven.

Proposition 1. *The Hessian submanifolds of M are embedded. Furthermore, there exists a thermodynamic representation that is a slice coordinate chart for any such submanifold.*

Conversely, if $(\mathcal{U}, (x^1, \dots, x^n))$ is a local thermodynamic representation on M , then $\iota^* dx^{r+1} = \dots = \iota^* dx^n = 0$ defines an embedded r -dimensional Hessian submanifold $\iota : N \rightarrow M$, with $r \leq \text{rank } \flat$, provided that $\partial_1 \Phi, \dots, \partial_r \Phi$ are independent for some local thermodynamic potential $\Phi \in C^\infty(\mathcal{U})$ [13].

Remark. As mentioned before, Ruppeiner claims that in the case a thermodynamic system has volume as a coordinate function of its entropy representation, the scalar curvature of the submanifold defined by constant volume yields information regarding microscopic interactions amongst the components of the macroscopic system [11]. It follows from the previous paragraph that such submanifold is not only Riemannian, but is also a particular Hessian submanifold of the space of equilibrium states. This raises the question, why should this Hessian submanifold (or any other) play a distinguished role? Adopting the common point of view of other geometrically-based physical theories, it follows that if scalar curvature yields any genuine physical information, it should not rely on the Hessian submanifold chosen. This idea offers a generalization of Ruppeiner's approach that can be applied to systems with no obvious analogue of volume (*e. g.* black hole solutions of General Relativity; *c. f.* Ref. [7]). Work in this direction is currently under progress.

Like any Hessian manifold, a Hessian submanifold ι of M admits a *dual Hessian structure*. Namely, if ∇ denotes the Levi-Civita connection of ι^*g , then $\bar{\nabla}^* := 2\nabla - \iota^*\bar{\nabla}$ is a flat connection which together with ι^*g forms a Hessian structure on N , called the *dual Hessian structure* of $(\iota^*g, \iota^*\bar{\nabla})$ [12]. This means that locally

$$\iota^*g = \bar{\nabla}^* d\Phi^*,$$

for some smooth local function Φ^* defined on N . If Φ is a local potential of g in M , such that $\iota^*\Phi$ shares domain \mathcal{V} with Φ^* , these two functions are related to each other through a Legendre transform [12]. To be specific, if $(\mathcal{U}, (x^1, \dots, x^n))$ is a thermodynamic representation and a slice coordinate system of N defined on the domain of Φ , then

$$\Phi^* = z^\alpha \frac{\partial \iota^*\Phi}{\partial z^\alpha} - \iota^*\Phi,$$

where $z^\alpha = \iota^*x^\alpha$, for all $\alpha \in \{1, \dots, r\}$. Since Φ is extensive, $\iota^*\Phi = \iota^*(x^j \partial_j \Phi)$, where j runs through $\{1, \dots, n\}$. Thus

$$\Phi^* = -\iota^* \left(x^A \frac{\partial \Phi}{\partial x^A} \right),$$

where A runs through $\{r+1, \dots, n\}$. It is important to remark that this approach offers a geometric (coordinate-free) portrait of total Legendre transforms. Furthermore, the fact that both $(\iota^*\bar{\nabla}, \iota^*g)$ and $(\bar{\nabla}^*, \iota^*g)$ form a Hessian structure may be translated as Ruppeiner's metrics' being invariant under *total* Legendre transforms.

Example 4. Consider \mathcal{E}_{ig} from Example 1. In this case, $\dim \ker b = 1$; 2-dimensional Hessian submanifolds of E are defined by $dU = 0$, $dV = 0$, and $dN = 0$, respectively.

It can readily be seen that the dual potentials at constant number of particles, at constant volume, and at constant temperature are proportional to the Gibbs free energy G , the grand potential Ω , and the internal energy, correspondingly (with $-T^{-1}$ as proportionality factor, where T denotes the temperature of the system) [13].

In the physics jargon, $1/T$ and p/T are referred to as the “natural variables” of G/T . Geometrically, this means that the former two produce an affine coordinate chart with respect to the connection which, together with ι^*g , constitutes a Hessian structure on the submanifold defined by constant number of particles. The function $-G/T$ is a global potential thereof. Analogous statements hold for the remaining submanifolds defined by $dU = 0$ and $dV = 0$.

It is important to remark that the above mentioned are not the only Hessian submanifolds of \mathcal{E}_{ig} . Any other submanifold defined by $\iota^*d(aU + bV + cN) = 0$, with $a, b, c \in \mathbb{R}$, at least one them different from zero, is also a Hessian submanifold.

4. CONCLUDING REMARKS

It was shown that a necessary and sufficient condition for a Hessian metric defined on a radiant manifold to be degenerate is that its local potentials be extensive (with respect to the radiant vector defined thereon). Since this is the case in classical thermodynamics, it is claimed that the straightforward geometric structure of equilibrium thermodynamics is formed on one hand by a degenerate Hessian metric (which comprises the Laws of thermodynamics), and a radiant structure (which embodies the concept of extensivity) on the other, which are compatible to each other in the sense that the local potentials of the former are extensive with respect to the latter. It is important to mention that extensivity has been approached in a more general fashion for self-gravitating thermodynamic systems by Belgiorno [1]. His point of view may be treated by considering extensivity with respect to affine vector fields [4], rather than radiant ones. The relationship between this more general notion of extensivity and Hessian structures is still unknown, and

its study, interesting on its own, might shed some light on the thermodynamic geometry of black holes. Research in this direction is currently under progress.

The Hessian submanifolds of degenerate Hessian manifolds are of particular physical interest due to their relationship to Ruppeiner's interaction hypothesis. They were shown to be embedded and to have thermodynamic representations as slice coordinates. This result justifies restricting the study of scalar curvature to manifolds defined by constant volume when relating it to interactions, as is commonly done. However, this also suggests that any physically relevant scalar should not depend on the choice of the Hessian submanifold.

ACKNOWLEDGEMENTS

The author thanks Professors G. F. Torres del Castillo and M. Montesinos for the encouraging discussions and the advice that helped to shape this paper. The author is also grateful to B. Díaz for his comments and suggestions on the manuscript, and to I. Rubalcava-García for her aid in the diffusion of this work. This work was partially supported by CONACyT, México, grant number 374393.

REFERENCES

- [1] F. Belgiorno, *Quasi-Homogeneous Thermodynamics and Black Holes*, J. Math. Phys. **44** (2003), no. 3, 1089–1128.
- [2] H. B. Callen, *Thermodynamics and an introduction to thermostatistics*, John Wiley & Sons, New York, 1985.
- [3] M. Á. García-Ariza, M. Montesinos, and G. F. Torres del Castillo, *Geometric thermodynamics: Black holes and the meaning of the scalar curvature*, Entropy **16** (2014), no. 12, 6515–6523.
- [4] W. Goldman, *Projective geometry on manifolds*, 2015.
- [5] W. Goldman and M. W. Hirsch, *Polynomial forms on affine manifolds*, Pacific J. Math **101** (1982), no. 1, 115–121.
- [6] ———, *The radiance obstruction and parallel forms on affine manifolds*, T. Am. Math. Soc. **286** (1984Dec), no. 2, 629–649.
- [7] D. Kubizňák and R. B. Mann, *Black Hole Chemistry*, 2014.
- [8] R. Mrugała, *On equivalence of two metrics in classical thermodynamics*, Phys. A **125** (1984), no. 2–3, 631–639.
- [9] G. Ruppeiner, *Thermodynamics: A riemannian geometric model*, Phys. Rev. A **20** (1979), no. 4, 1608–1613.
- [10] ———, *Riemannian geometric approach to critical points: General theory*, Phys. Rev. E **57** (1998May), no. 5, 5135–5145.
- [11] ———, *Thermodynamic curvature measures interactions*, Am. J. Phys **78** (2010), 1170–1180, available at [arXiv:1007.2160](https://arxiv.org/abs/1007.2160).
- [12] H. Shima, *The geometry of hessian structures*, World Scientific, New Jersey, 2007.
- [13] G. F. Torres del Castillo and M. Montesinos Velásquez, *Riemannian structure of the thermodynamic phase space*, Rev. Mexicana Fís. **39** (1993), no. 2, 194–202.
- [14] F. Weinhold, *Metric geometry of equilibrium thermodynamics*, J. Chem. Phys. **63** (1975), no. 6, 2479–2483.

FACULTAD DE CIENCIAS FÍSICO MATEMÁTICAS, BENEMÉRITA UNIVERSIDAD AUTÓNOMA DE PUEBLA,
APARTADO POSTAL 1152, 72000 PUEBLA, MEXICO

E-mail address: magarciaariza@alumnos.fcfm.buap.mx